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Linear Quadratic Regulator. (LQR)

- Recall: $\dot{x} = Ax + Bu$
  $u = -kx$

- We have seen the pole placement method for k:

$$k = ? \ : \ eig(A - Bk) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$ → desired eig values.

  - We force the desired performance onto the system

Now, using LQR:

- We don't care about the position of the poles (only that they are in LHP):

  k = ? such that (A-Bk) is stable.

  And minimizing a certain cost function. } ex. minimize energy.

  (optimal control).

→ Different situations call for different design methods.

- For LQR we are still designing 'k' but the priorities are different. (Q - quadratic contains the cost function)

Preliminaries

- A real, symmetric, square matrix can be classified as:
  i) Positive Definite if all of its eigenvalues are positive.
  ii) Positive Semi-Definite if all of its eig. values are ≥ 0
  iii) Negative Definite " " < 0
  iv) Negative Semidefinite " " ≤ 0
  v) indefinite if none of the above.

$P = P^T > 0$ } Notation for positive definite.

- If $P = P^T > 0$ then $x^T P x > 0$, $\forall x \neq 0$, $x \in \mathbb{R}$
- if $P = P^T < 0 \implies x^T P x < 0$, $\forall x \neq 0$

Ex.

$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \longrightarrow$ Positive definite

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow x^T P x = x_1 x_2 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= x_1^2 + 2x^2 > 0$

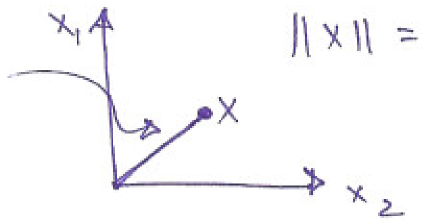
$\forall x_1, x_2 \neq 0$

- Energy in a vector function $f(t)$ is given by:

$$\boxed{E = \int_0^\infty f^T(t) f(t)\, dt}$$

Signal Norms. (Recall, norm is the size of a vector) } Numerical Analysis.

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



$\|x\| = \sqrt{x_1^2 + x_2^2} = \sqrt{x^T x}$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \sqrt{x^T x} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$

} euclidian norm

- If x is moving in space then $\|x\| = \sqrt{x^T x}$ represents the energy at a specific moment in time.

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Signal Norms Cont'd

$X \in \mathbb{R}^n$, $\|X\| = \sqrt{X^T X}$ } Euclidean Norm.

$$L_2 \text{ norm} = \sqrt{\int_0^\infty \|X(t)\|^2 dt} = \|X\|_2$$

$\|X\|^2 = X^T X$ ⟶ energy of signal x @ time t.

$\|X\|_2^2 = \int_0^\infty X^T(t)\, X(t)\, dt$ ⟶ total energy of signal from $t = 0 \to \infty$.

- Occassionally, we are interested in the size of a linear combination of the components of a signal $x(t)$, say $Mx(t)$ where M is an appropriate matrix.

$$\|Mx\|_2^2 = \int_0^\infty (Mx)^T Mx\, dt = \int_0^\infty x^T \underbrace{(M^T M)}_{Q} x\, dt$$

$$= \int_0^\infty x^T Q x\, dt$$

Ex.

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ⟶ suppose we want to minimize the energy given to variable x_2

$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore$ $Mx = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

chosen value.

$$\|Mx\|_2^2 = \int_0^\infty x^T Q x\, dt \qquad M^T M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \int_0^\infty x^T M^T M x\, dt = \int_0^\infty x_1^2\, dt$$

- The LQR consists of finding K $(u=-Kx)$ to minimize the quadratic cost function.

$$\boxed{J=\int_0^\infty (x^TQx + u^TRu)\,dt}$$

energy in control

energy in states.

$Q \geq 0 \quad R > 0$

- In matlab: $K = LQR(A,B,Q,R)$ } K will be designed such that J is minimum.

- Large R means that the controller consumes more energy

- Large Q means the system consumes more energy.

Consider:

control energy

$$J = \|My\|_2^2 + \|Nu\|_2^2$$

output energy.

$$J = \|MCx\|_2^2 + \|Nu\|_2^2 = \int_0^\infty \|MCx\|^2 + \|Nu\|^2\,dt$$

$$= \int_0^\infty (x^T \underbrace{C^TM^TMC}_{Q} x + u^T \underbrace{N^TN}_{R} u)\,dt$$

$$= \int_0^\infty (x^TQx + u^TRu)\,dt \quad \Big\} \; R>0, \quad Q = C^TM^TMC$$

*Conditions for the existence of a solution:

1) $R>0$, $Q \geq 0$

2) the pair (A,B) is controllable. (stabilizable) → uncontrollable part is stable.

3) the pair (A, MC) is observable (detectable).

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Algorithm for Minimization.

① Form the Ricatti equation:

$$\boxed{PA + A^TP - PBR^{-1}B^TP + Q = 0} \quad (1)$$

- applies to the case where upper limit of t is ∞.

② Solve (1) for a positive definite & symmetric matrix 'P'.

② Then, $\boxed{K = R^{-1}B^TP}$ & $\boxed{u = -Kx}$

Result.

* $A-BK$ will be stable
* minimum value of J is given by $x^T(0)Px(0)$

Matlab: $[K, P, E] = LQR(A, B, Q, R)$

- E → c.L. eigenvalues.
- P → solution of (1)
- K → feedback gain.